

Chapter 1

POINT KINETICS

1.1 INTRODUCTION

The Point Kinetics Model is widely used in reactor dynamics because of the apparent simplicity of the resulting equations. The main difficulty of this method lies in obtaining the necessary parameters describing the reactor. Nevertheless, many characteristics of the dynamic behaviour of a nuclear reactor can be deduced from them. Also, the point kinetics method can be used as a basic test of the more sophisticated methods used in full space-time calculations; if a given method is not able to pass the test on a reactor considered as a single point or region, it will have difficulties when considering the reactor with many regions.

We present here a very sketchy demonstration of point kinetics; another course will be devoted to this method.

1.2 DERIVATION

The starting point of the point kinetics equation is of course the space-time neutron transport equations. This transport equation is almost impossible to solve, even on the most powerful computers available today. We resort instead to the diffusion approximation of this, which is obtained after applying the P1 spherical harmonics approximation to the directional flux. This

space time diffusion equation is

$$\frac{1}{v_g} \frac{\partial}{\partial t} \Phi_g(\vec{r}, t) = -\nabla \cdot \vec{J}_g(\vec{r}, t) - \Sigma_{tg} \Phi_g + \sum_{g'=1}^G \Sigma_{sg'} \Phi_{g'} + \chi_g^p (1 - \beta) \sum_{g'=1}^G \nu \Sigma_{fg'} \Phi_{g'} + \sum_{i=1}^D \chi_{ig}^d \lambda_i C_i(\vec{r}, t) \quad (1.1)$$

$$\vec{J}_g = -D_g \vec{\nabla} \Phi_g \quad (1.2)$$

$$\frac{\partial}{\partial t} C_i(\vec{r}, t) = \beta_i \sum_{g'=1}^G \nu \Sigma_{fg'} \Phi_{g'} - \lambda_i C_i(\vec{r}, t) \quad (1.3)$$

$g = 1, \dots, G$ and $i = 1, \dots, D$

The main idea in point kinetics is to separate the (multigroup) neutron flux in a part depending only on space, and another depending only on time, in the following way,

$$[\Phi(\vec{r}, t)] = [S(\vec{r}, t)]T(t) \quad (1.4)$$

This equation is still exact, but $[S(\vec{r}, t)]$ depends on both space and time in this approach.

We now introduce a weight vector

$$[W(\vec{r})] = \begin{bmatrix} W_1(\vec{r}) \\ W_2(\vec{r}) \\ \vdots \\ W_G(\vec{r}) \end{bmatrix} \quad (1.5)$$

whose role will be to obtain completely general expressions.

There is a degree of arbitrariness in the choice of $T(t)$ and $[S(\vec{r}, t)]$. Only the product of the two really counts. We thus introduce a normalization constraint on $T(t)$ and $[S(\vec{r}, t)]$, We define

$$T(t) = \langle [W(\vec{r})]^T [v]^{-1} [\phi] \rangle \quad (1.6)$$

and it follows that after 1.4, $[S(\vec{r}, t)]$ must obey the constraint

$$\langle [W(\vec{r})]^T [v]^{-1} [S(\vec{r}, t)] \rangle = 1 \quad (1.7)$$

where the symbol $\langle \rangle$ implies integration over the whole spatial extent of the reactor. The factor $[S(\vec{r}, t)]$ becomes the "form function", whereas $T(t)$ is the

amplitude function. Remark that $T(t)$ represents loosely the total number of neutrons in the reactor, this number depending somewhat on the chosen weight vector. The constraint on $[S(\vec{r}, t)]$ does not depend on time; the form function may vary in time, but it's spatial integral is time independent. Thus $T(t)$ represents by itself the time dependence of the neutron population.

We can now obtain a differential equation for $T(t)$ by substitution of $[S(\vec{r}, t)] T(t)$ for $\phi(\vec{r}, t)$ in the space time diffusion equations that have been pre-multiplied $[W(\vec{r})]^T$ and by integrating over the whole reactor core.

We define the quantities

$$C_i(t) = \frac{\langle [W]^T [\chi_i^d] C_i \rangle}{\langle [W]^T [v]^{-1} [S] \rangle} \quad (1.8)$$

$$\Lambda(t) = \frac{\langle [W]^T [v]^{-1} [S] \rangle}{\langle [W]^T \left\{ (1 - \beta) [\chi^p] + \sum_{i=1}^D \beta_i [\chi_i^d] \right\} [\nu \Sigma_f]^T [S] \rangle} \quad (1.9)$$

$$\beta_i(t) = \beta_i \frac{\langle [W]^T [\chi_i^d] [\nu \Sigma_f]^T [S] \rangle}{\langle [W]^T \left\{ (1 - \beta) [\chi^p] + \sum_{i=1}^D \beta_i [\chi_i^d] \right\} [\nu \Sigma_f]^T [S] \rangle} \quad (1.10)$$

and

$$\beta(t) = \sum_{i=1}^D \beta_i(t) \quad (1.11)$$

$$\rho(t) = \frac{\langle [W]^T \left\{ \nabla \cdot [D] \nabla [S] - [\Sigma] [S] + \left((1 - \beta) [\chi^p] + \sum_{i=1}^D \beta_i [\chi_i^d] \right) [\nu \Sigma_f]^T [S] \right\} \rangle}{\langle [W]^T \left\{ (1 - \beta) [\chi^p] + \sum_{i=1}^D \beta_i [\chi_i^d] \right\} [\nu \Sigma_f]^T [S] \rangle} \quad (1.12)$$

With these definitions, the space-time kinetics equations become

$$\frac{dT(t)}{dt} = \frac{\rho(t) - \beta(t)}{\Lambda(t)} T(t) + \sum_{i=1}^D \lambda_i C_i(t) \quad (1.13)$$

and

$$\frac{dC_i(t)}{dt} = \frac{\beta_i(t)}{\Lambda(t)} T(t) - \lambda_i C_i(t) \quad (1.14)$$

which are the point kinetics equations.

1.3 THE POINT KINETICS APPROXIMATION

So far no approximations have been made. However the calculation of the parameters $\rho(t)$, $\beta_i(t)$ and $\Lambda(t)$ depend, by definition on the form function $[S(\vec{r}, t)]$. In order to know $[S(\vec{r}, t)]$ implies in turn knowing the neutron flux $[\phi(\vec{r}, t)]$ which necessitates a complete solution of the space time equations. Thus it becomes very difficult to determine the point kinetics parameters.

The way out of this consists in replacing $[S(\vec{r}, t)]$ by a function depending on space alone, denoted $[S(\vec{r})]$. This function is usually provided by the static, initial neutron distribution, before perturbations have been applied. In this case, it appears clearly that the resulting parameters are necessarily approximative.

For instance, the β_i and Λ lose their time dependence. As for the reactivity $\rho(t)$, its value depends on the time variations of nuclear cross-sections and diffusion coefficients. But these parameters are applied to a form function which does not correspond to the instantaneous state of the core during the transient. In this case, it can be shown that the best choice for the weight function $[W]^T$ is the adjoint flux of the initial neutron flux. Any other choice would increase the error on $\rho(t)$, and therefore increase the error on the amplitude $T(t)$.

Despite all these difficulties, point kinetics is still the most widely used method in reactor kinetics. This is mainly due to the small number of equations to solve, and to the fact that only one spatial calculation (initial state of the core) is necessary. However, despite this apparent simplicity, many difficulties are hidden behind these equations.

1.4 AN ANALYTIC SOLUTION

In the presence of specified reactivity changes, with all other parameters fixed, it is possible to obtain analytic solutions to the point kinetic equations. The mathematical analysis of these is outside the scope of this course. We outline briefly the procedure here, for a step change of reactivity. Generally speaking, ρ varies in complicated ways as a function of time. We chose time intervals within which ρ stays relatively constant. This is the “stairway” approximation to the function $\rho(t)$. Reactivity thus stays constant within a time step.

We define a vector

$$[\psi] = \begin{bmatrix} T(t) \\ C_1(t) \\ C_2(t) \\ \vdots \\ C_D(t) \end{bmatrix} \quad (1.15)$$

and the following matrix

$$[R] = \begin{bmatrix} \frac{\rho - \beta}{\Lambda} & \lambda_1 & \lambda_2 & \dots & \lambda_D \\ \frac{\beta_1}{\Lambda} & -\lambda_1 & & & \\ \frac{\beta_2}{\Lambda} & & -\lambda_2 & & \\ \vdots & & & \ddots & \\ \frac{\beta_D}{\Lambda} & & & & -\lambda_D \end{bmatrix} \quad (1.16)$$

so that the system 1.13 and 1.14 becomes

$$\frac{d[\psi]}{dt} = [R][\psi] \quad (1.17)$$

We also introduce a vector $[\psi']$ related to the vector $[\psi]$ by application of a linear transformation $[T]$,

$$[\psi] = [T][\psi'] \quad (1.18)$$

The system 1.17 then becomes, after substitution of 1.18,

$$\frac{d}{dt} [\psi] = [R][T][\psi'] \quad (1.19)$$

or

$$[T] \frac{d}{dt} [\psi'] = [R][T][\psi'] \quad (1.20)$$

which becomes

$$\frac{d}{dt} [\psi'] = [T]^{-1} [R][T][\psi'] \quad (1.21)$$

The operator $[T]$ is chosen so that it diagonalizes the system 1.21, we will get

$$\frac{d}{dt} [\psi'] = [\mathcal{D}][\psi'] \quad (1.22)$$

where $[\mathcal{D}]$ is a diagonal matrix. The individual elements of $[\psi']$ are easy to calculate in this new basis, and are simply

$$[\psi'(t)] = \exp(\omega_i(t - t_0)) [\psi'(t_0)] \quad (1.23)$$

and, after the inverse transformation, we get

$$[T]^{-1} [\psi] = \exp (\omega_i (t - t_0)) [T]^{-1} [\psi (t_0)] \quad (1.24)$$

or

$$[\psi (t)] = [T] \exp (\omega_i (t - t_0)) [T]^{-1} [\psi (t_0)] \quad (1.25)$$

We are left with calculating the elements of the matrix $[T]$, and the ω_i . To determine them, let us consider the systems 1.21 and 1.22, which give

$$[T]^{-1} [R] [T] = [D] \quad (1.26)$$

A result of linear analysis ¹ shows that the elements of the diagonal matrix $[D]$ are the eigenvalues of the matrix $[R]$, and that the columns of the matrix $[T]$ are made of the corresponding eigenvectors. The elements of the matrix $[T]^{-1}$ can be calculated by direct inversion, or by solving the adjoint problem. Thus, we have to solve the eigenvalue-eigenvector problem of the matrix $[R]$.

The eigenvalues are given by

$$\det ([R - \omega_i I]) = 0 \quad (1.27)$$

The determinant expansion, along the first column of the matrix $[R - \omega_i I]$ will give

$$\left\{ \left(\frac{\rho - \beta}{\Lambda} - \omega \right) + \sum_{i=1}^D \frac{\beta_i \lambda_i}{\Lambda (\lambda_i + \omega)} \right\} \prod_{j=1}^D (-\lambda_j - \omega) = 0 \quad (1.28)$$

Note that $\omega = \lambda_i$ is not a solution of 1.28. Rather they are solutions of

$$\left(\frac{\rho - \beta}{\Lambda} - \omega \right) + \sum_{i=1}^D \frac{\beta_i \lambda_i}{\Lambda (\lambda_i + \omega)} = 0 \quad (1.29)$$

This is known as Nordheim equations. The zeroes of this equation can only be obtained approximately (method of Newton and variants...) when more than three delayed neutron groups are involved. After obtaining the eigenvalues, we get the eigenvectors

$$[U]^i = \begin{bmatrix} u_0^i \\ u_1^i \\ u_2^i \\ \vdots \\ u_D^i \end{bmatrix} \quad (1.30)$$

¹see for example J. H. Wilkinson, The Algebraic Eigenvalue Problem (Oxford University Press, Oxford, 1965).

where the index i is used to show that the eigenvector $[U]^i$ belongs to the eigenvalue ω_i . We get

$$[U]^i = \begin{bmatrix} 1 \\ u_1^i \\ u_2^i \\ \vdots \\ u_D^i \end{bmatrix} \quad (1.31)$$

To find the remaining elements, we solve

$$[R - \omega_i I][U]^i = 0 \quad (1.32)$$

or, more directly,

$$\begin{bmatrix} \frac{\rho-\beta}{\Lambda} - \omega_i & \lambda_1 & \lambda_2 & \cdot & \lambda_D \\ \frac{\beta_1}{\Lambda} & -\lambda_1 - \omega_i & & & \\ \frac{\beta_2}{\Lambda} & & -\lambda_2 - \omega_i & & \\ \vdots & & & \ddots & \\ \frac{\beta_D}{\Lambda} & & & & -\lambda_D - \omega_i \end{bmatrix} \begin{bmatrix} 1 \\ u_1^i \\ u_2^i \\ \vdots \\ u_D^i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (1.33)$$

We then get

$$u_j^i = \frac{\beta_j}{\Lambda} \cdot \frac{1}{\lambda_j + \omega_i} \quad (1.34)$$

and the matrix $[T]$ can be constructed

$$[T] = \begin{bmatrix} 1 & 1 & \cdot & \cdot & 1 \\ \frac{\beta_1}{\Lambda} \cdot \frac{1}{\lambda_1 + \omega_0} & \frac{\beta_1}{\Lambda} \cdot \frac{1}{\lambda_1 + \omega_1} & \cdot & \cdot & \frac{\beta_1}{\Lambda} \cdot \frac{1}{\lambda_1 + \omega_D} \\ \frac{\beta_2}{\Lambda} \cdot \frac{1}{\lambda_2 + \omega_0} & \frac{\beta_2}{\Lambda} \cdot \frac{1}{\lambda_2 + \omega_1} & \cdot & \cdot & \frac{\beta_2}{\Lambda} \cdot \frac{1}{\lambda_2 + \omega_D} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \frac{\beta_D}{\Lambda} \cdot \frac{1}{\lambda_D + \omega_0} & \frac{\beta_D}{\Lambda} \cdot \frac{1}{\lambda_D + \omega_1} & \cdot & \cdot & \frac{\beta_D}{\Lambda} \cdot \frac{1}{\lambda_D + \omega_D} \end{bmatrix} \quad (1.35)$$

The determination of the ω_i shows that these differ much from each other, the minimum being about, $\frac{\rho-\beta}{\Lambda}$, whereas the maximum is greater than $-\lambda_1$ in all cases, and greater than 0 if $\rho > 0$. This characterizes the point kinetics equations as being "stiff".

We conclude over the following observations:

- 1 * With more than one group of delayed neutrons, the point kinetics equations are difficult to solve.

- 2 ° It is impossible to get a true analytical solution, because the transcendental equation for the ω_i , which can be solved only by numerical methods.
- 3 ° The matrix $[T]$ must be inverted, or the adjoint problem must be solved.
- 4 ° As ρ varies in time, all this work must be redone at each time step.

Because of all this, the analytic solution of the point kinetics equations is too difficult to be a practical method. Rather, numerical methods will be used in place of the analytic method.

1.5 NUMERICAL SOLUTION

We examine here a few simple numerical methods to solve the point kinetics equations. A full course on numerical methods could be given on this topic alone. We restrain ourselves to the simplest practical methods, mainly because they can be applied to the full space time kinetics with little modifications. Also, some important conclusions can be derived from even the simplest methods, even with a little mathematical rigor.

In practice, the choice of a numerical is based on three criteria,

- truncation error
- stability
- calculation effort

Proper equilibrium between these three is chosen on individual needs. For example, if many situations of very different states are to be calculated, the stability criterion might be the most important one.

In order to review each of the three criteria for a few very simple methods, we write the point kinetics equations in the form

$$\frac{d}{dt}\Psi = R\Psi \quad (1.36)$$

In this section, we use the following notation:

- $\Delta t = t_{n+1} - t_n$

- $\Psi^n \equiv \Psi(t_n)$

Furthermore, in the stability analysis, we suppose that the operator R stays constant in any given transient. We use the principle, without proving it, that the stability properties of a given numerical scheme do not depend on the particular basis used to represent the resultant vectors and matrices.

In this section, we will consider only two numerical schemes, the explicit method and the implicit method. The first step involved in these methods is to replace the temporal derivatives in 1.36 by

$$\frac{d}{dt}\Psi \approx \frac{\Psi^{n+1} - \Psi^n}{\Delta t} \quad (1.37)$$

1.5.1 Explicit Method

The explicit method consists in replace the right hand side term in 1.36 by $R\Psi^n$. In this case, 1.36 becomes

$$\Psi^{n+1} = (I + R\Delta t)\Psi^n \quad (1.38)$$

Truncation Error

Formally, the exact solution of the differential equation 1.36 is an exponential of the matrix R ,

$$\Psi^{n+1} = \exp(R\Delta t)\Psi^n \quad (1.39)$$

The truncation error is the difference between the approximate solution and the exact solution. Consequently, the truncation error ET is given by

$$ET = (I + R\Delta t - \exp(R\Delta t)) \quad (1.40)$$

If we expand the matrix exponential,

$$ET = (I + R\Delta t - (I + R\Delta t + \frac{1}{2}R^2\Delta t^2 + \dots)) \quad (1.41)$$

and

$$ET \approx -\frac{1}{2}R^2\Delta t^2 \quad (1.42)$$

The explicit method is thus of order Δt^2 .

Stability

In order to examine the stability of the explicit method, we write the differential equation in the basis that diagonalizes the matrix R . In this case, we go back to equation 1.38, which we rewrite in the new basis

$$x^{n+1} = (1 + \omega \Delta t)x^n \quad (1.43)$$

The ω will be the eigenvalues of R , this is to say the roots of the Nordheim equation 1.28.

Starting from the initial conditions x^0 at time $t = 0$, the first cycle of calculation will give rise to the vector x^1 ,

$$x^1 = (1 + \omega \Delta t)x^0 \quad (1.44)$$

The application of the same operator on x^1 will give rise to the second cycle

$$x^2 = (1 + \omega \Delta t)x^1 \quad (1.45)$$

Thus

$$\begin{aligned} x^2 &= (1 + \omega \Delta t)(1 + \omega \Delta t)x^0 \\ &= (1 + \omega \Delta t)^2 x^0 \end{aligned} \quad (1.46)$$

Continuing this process, we find

$$x^{n+1} = (1 + \omega \Delta t)^{n+1} x^0 \quad (1.47)$$

For example, if we had a negative reactivity, we know that all the ω are negative, and that we must have a decreasing solution. Therefore,

$$|1 + \omega \Delta t| < 1 \quad (1.48)$$

which can be written

$$-1 < 1 + \omega \Delta t < 1 \quad (1.49)$$

We replace ω by $-|\omega|$ to emphasize that all ω are negative.

$$-1 < 1 - |\omega| \Delta t < 1 \quad (1.50)$$

The system will be stable if these two inequalities are met. We must examine the two possibilities : $1 - |\omega| \Delta t < 1$ and $-1 < 1 - |\omega| \Delta t$

Case 1 : $1 - |\omega| \Delta t < 1$

We subtract 1 on each side if this inequality, to get

$$-|\omega| \Delta t < 0 \quad (1.51)$$

which is always true.

Case 2 : $-1 < 1 - |\omega| \Delta t$

We subtract 1 on each side of the inequality,

$$-2 < -|\omega| \Delta t \quad (1.52)$$

which is

$$-|\omega| \Delta t > -2 \quad (1.53)$$

We change the sign by multiplying by -1. We must change the $>$ by a $<$

$$|\omega| \Delta t < 2 \quad (1.54)$$

or finally

$$\Delta t < \frac{2}{|\omega|} \quad (1.55)$$

There is thus a condition on Δt to insure stability of the explicit scheme.

This stability condition 1.55 is very restrictive. The negative roots of the Nordheim equations are very negative, and the most negative one is inferior to $-\lambda_D$, being around $-\frac{\beta}{\Lambda}$. Time steps lower than 10 or 20 milliseconds will be necessary to insure stability.

Calculation Effort

The explicit method, given in 1.38 needs only a multiplication of the matrix $\Delta t R$ by the vector Ψ^n to get the solution at the next time step. From this point of view, the explicit method is very simple and requires very little calculational effort to put in place.

1.5.2 Implicit Method

In the implicit scheme, we replace the right hand side of 1.36 by $R\Psi^{n+1}$, (instead of $R\Psi^n$ as in the explicit method). Thus

$$\frac{\Psi^{n+1} - \Psi^n}{\Delta t} = R\Psi^{n+1} \quad (1.56)$$

We group terms in Ψ^{n+1} and in Ψ^n together

$$(I - \Delta t R)\Psi^{n+1} = \Psi^n \quad (1.57)$$

and writing Ψ^{n+1} in terms of Ψ^n ,

$$\Psi^{n+1} = (I - \Delta t R)^{-1}\Psi^n \quad (1.58)$$

Truncation Error

The truncation error will be

$$ET = \Psi^{n+1} - \exp(R\Delta t)\Psi^n \quad (1.59)$$

and in terms of the implicit scheme,

$$ET = ((I - \Delta t R)^{-1} - \exp(R\Delta t))\Psi^n \quad (1.60)$$

Making the expansions

$$(I - \Delta t R)^{-1} = I + \Delta t R + (\Delta t R)^2 + \dots \quad (1.61)$$

and

$$\exp(R\Delta t) = I + \Delta t R + \frac{1}{2}(R\Delta t)^2 + \dots \quad (1.62)$$

which we substitute in the truncation error expression

$$ET = (I + \Delta t R + (\Delta t R)^2 + \dots) - (I + \Delta t R + \frac{1}{2}(R\Delta t)^2 + \dots)\Psi^n \quad (1.63)$$

Stopping the expansion to terms in $(\Delta t R)^2$

$$ET \approx (I + \Delta t R + (\Delta t R)^2) - (I + \Delta t R + \frac{1}{2}(R\Delta t)^2)\Psi^n \quad (1.64)$$

and finally,

$$ET \approx -\frac{1}{2}(R\Delta t)^2\Psi^n \quad (1.65)$$

Consequently, the implicit method is a method of order Δt^2 . Also, the coefficient of Δt^2 is $\frac{1}{2}$, the same as that of the explicit method.

Stability

Again we use the basis that **diagonalizes** the system. We find

$$x^1 = (I - \omega \Delta t)^{-1} x^0 \quad (1.66)$$

$$\begin{aligned} x^2 &= (I - \omega \Delta t)^{-1} x^1 \\ x^2 &= (I - \omega \Delta t)^{-2} x^0 \end{aligned} \quad (1.67)$$

and after $n + 1$ intervals the solution vector is

$$x^{n+1} = (I - \omega \Delta t)^{-(n+1)} x^0 \quad (1.68)$$

We will have a stable solution if

$$-1 < \frac{1}{1 - \omega \Delta t} < 1 \quad (1.69)$$

We must have a method whose solution tends towards 0 when reactivity is negative, ie when all ω are negative. in this case, $-\omega$ can be replaced by $|\omega|$, which gives

$$-1 < \frac{1}{1 + |\omega| \Delta t} < 1 \quad (1.70)$$

As $1 + |\omega| \Delta t$ is positive, we can multiply each member of the previous inequality by this factor without changing the inequalities themselves,

$$-(1 + |\omega| \Delta t) < 1 < (1 + |\omega| \Delta t) \quad (1.71)$$

We examine in turn each of them:

Case 1 : $-(1 + |\omega| \Delta t) < 1$

Subtracting 1 on each side gives

$$-|\omega| \Delta t < 0 \quad (1.72)$$

But the absolute value of $|\omega| \Delta t$ is always positive because Δt is always positive. Then $-|\omega| \Delta t$ is always negative, and the inequality is always true.

Case 2 : $1 < (1 + |\omega| \Delta t)$

Subtracting 1 on each side gives

$$0 < |\omega| \Delta t \quad (1.73)$$

or equivalently

$$|\omega| \Delta t > 0 \quad (1.74)$$

But the value of $|\omega| \Delta t$ is **always positive**, because Δt is always positive. The inequality is thus **always true**.

The conditions on stability are **thus always** verified, and we can conclude that the implicit method is **unconditionally stable**.

Calculation Effort

The implicit method, given in 1.58 **requires the** inversion of the $\Delta t R$ matrix to get the solution at the next time step. This matrix can change as the kinetics parameters, especially **the reactivity**, change during the transient. The inversion will have to be **performed each** time. From this point of view, the implicit method is much more **computationally** intensive than the explicit method. But it has the advantage of **unconditional** stability, at the price of a matrix inversion.

Equation (6.3.8) shows that $S(\infty)\sigma_{a1}^{Sm}$, the asymptotic one-group absorption cross section for Sm^{149} , is γ^{Nd} (1.13 percent) times the one-group fission cross section. Thus γ^{Nd} gives a measure of the competition for neutrons between the fission process and absorption in Sm^{149} once an equilibrium concentration has been attained. Moreover we see that this equilibrium concentration is independent of the flux level Φ_1 . Thus, after a reactor has been operating at high power for a month or so, an equilibrium of Sm^{149} is attained and remains in competition for neutrons for the rest of the lifetime of the core.

For a low-power reactor the same potential competition exists. However it takes so much longer for the Sm^{149} concentration to reach equilibrium that, in many cases, its effect throughout lifetime is negligible. This fact can be seen from (6.3.6), which shows that, if $\sigma_{a1}^{Sm}\Phi_1 \ll \lambda^{Pm}$ and $S(0) = 0$,

$$S(t) = \frac{\gamma^{Nd}\Sigma_{f1}}{\sigma_{a1}} [1 - \exp(-\sigma_{a1}^{Sm}\Phi_1 t)]. \quad (6.3.9)$$

If Φ_1 is, say, 10^8 neutrons/cm²/sec, $\exp(-\sigma_{a1}^{Sm}\Phi_1 t)$ will be close to unity until t exceeds 10^9 seconds (about 13 years), at which time it will be approximately 0.996.

After Shutdown from Equilibrium High-Power Operation. If equilibrium conditions have been attained and the reactor has then been shut down, the time behavior of Pm^{149} and Sm^{149} is given by (6.3.5) and (6.3.6) with $\Phi_1 = 0$ and $P(0)$ and $S(0)$ replaced by the equilibrium values (6.3.7) and (6.3.8). The result is

$$\begin{aligned} P(t) &= \frac{\gamma^{Nd}\Sigma_{f1}\Phi_1(0)}{\lambda^{Pm}} \exp(-\lambda^{Pm}t), \\ S(t) &= \frac{\gamma^{Nd}\Sigma_{f1}}{\sigma_{a1}^{Sm}} + \frac{\gamma^{Nd}\Sigma_{f1}\Phi_1(0)}{\lambda^{Pm}} [1 - \exp(-\lambda^{Pm}t)], \end{aligned} \quad (6.3.10)$$

where t is now the time after shutdown and $\Phi_1(0)$ is the flux prior to shutdown (i.e., the flux associated with the initial equilibrium concentrations of Pm^{149} and Sm^{149}).

Physically these equations state that the equilibrium samarium present at shutdown, $\gamma^{Nd}\Sigma_{f1}/\sigma_{a1}^{Sm}$, is augmented by an amount $\gamma^{Nd}\Sigma_{f1}\Phi_1(0)/\lambda^{Pm}$ as the equilibrium promethium decays into samarium. This final concentration *does* depend on the value of Φ_1 prior to shutdown, and the increase in concentration will be significant for

$$\frac{\lambda^{Pm}}{\Phi_1} = \frac{3.5 \times 10^{-6}}{\Phi_1} < \sigma_{a1}^{Sm} = 4 \times 10^{-20}.$$

Thus, when Φ_1 is slightly less than 10^{14} , the concentration of Sm^{149} present after shutdown is about twice its equilibrium, full-power concentration. After the reactor is turned on again, the extra samarium will burn out and the equilibrium conditions (6.3.7) and

FROM A. F. HENRY, NUCLEAR REACTOR ANALYSIS, MIT PRESS, 1975.

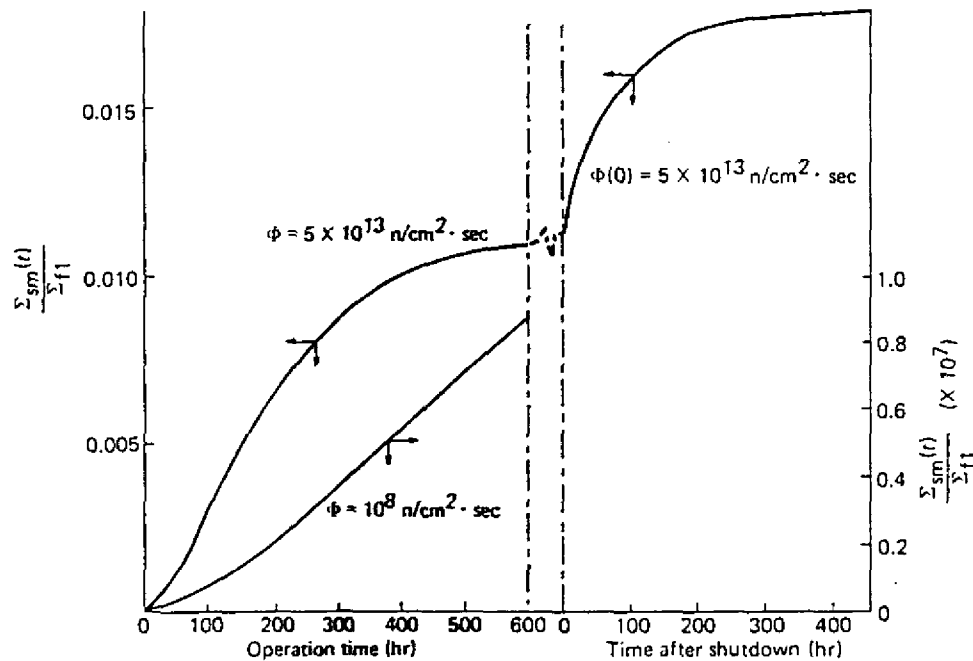


Figure 6.4 Behavior of Sm^{149} in several transient situations.

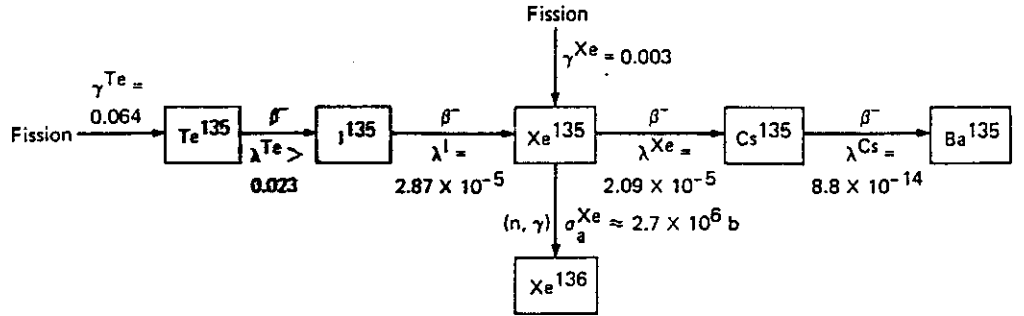
(6.3.8) will return. Thus, provided the reactor contains enough excess fuel to go critical despite the increased absorption in Sm^{149} after shutdown, that extra absorption produces no lasting effects.

Figure 6.4 shows the behavior of Sm^{149} under several transient conditions.

The Xe^{135} Fission-Product Chain

From the viewpoint of criticality and control, the isotope Xe^{135} is the most important of all the fission products. It has a large absorption resonance that peaks at $E = 0.082$ eV and results in an absorption cross section of approximately 2.7×10^6 barns at 0.025 eV. Xenon-135 is formed from the decay of iodine-135 (6.7-hour half-life) and is itself radioactive (9.2-hour half-life). It is part of the fission chain shown in Figure 6.5, where all the decay constants λ^i are in sec^{-1} and the fission fractions γ^{Te} and γ^{Xe} are those appropriate to thermal fission of U^{235} . None of the absorption cross sections in the chain except that of Xe^{135} are large enough to be of any significance.

The decay of Te^{135} is so fast, and that of Cs^{135} is so slow (2.6-million-year half-life), that we may assume for our purposes that I^{135} is formed directly from fission with a yield $\gamma^{\text{I}} = 0.064$ and that the chain ends with the destruction, by β decay or neutron


 Figure 6.5 The Xe^{135} fission-product chain.

absorption, of Xe^{135} . Thus, if $I(t)$ and $X(t)$ represent the concentrations of I^{135} and Xe^{135} at some location $\mathbf{r}$ in the reactor, the equations specifying the time dependence of these concentrations may be written (with the $\mathbf{r}$ dependence suppressed),

$$\begin{aligned} \frac{\partial I(t)}{\partial t} &= \gamma^I \Sigma_{f1} \Phi_1(t) - \lambda^I I(t), \\ \frac{\partial X(t)}{\partial t} &= \gamma^{Xe} \Sigma_{f1} \Phi_1(t) + \lambda^I I(t) - [\sigma_{a1}^{Xe} \Phi_1(t) + \lambda^{Xe}] X(t), \end{aligned} \quad (6.3.11)$$

where Σ_{f1} , $\Phi_1(t)$, and σ_{a1}^{Xe} are the one-group, macroscopic fission cross section, flux, and microscopic absorption cross section for Xe^{135} . As with the samarium chain, we shall think of the yields γ^I and γ^{Xe} as representing averages over the fissionable isotopes present at location $\mathbf{r}$ and over the energy spectrum of the neutrons causing fission at that point. Strictly speaking Σ_{f1} and σ_{a1}^{Xe} are also time-dependent. However, in the time scale (tens of hours) of xenon transients, this time dependence may be neglected. (Note, however, that changes of σ_{a1}^{Xe} with time due to changes in temperature may have a short-term effect on the criticality of the reactor. We shall deal with such matters when we consider reactor kinetics.)

For $\Phi_1(t)$ constant the solution of (6.3.11) is

$$\begin{aligned} I(t) &= \frac{\gamma^I \Sigma_{f1} \Phi_1}{\lambda^I} [1 - \exp(-\lambda^I t)] + I(0) \exp(-\lambda^I t), \\ X(t) &= X(0) \exp(-(\sigma_{a1}^{Xe} \Phi_1 + \lambda^{Xe})t) + \frac{\gamma \Sigma_{f1} \Phi_1}{\sigma_{a1}^{Xe} \Phi_1 + \lambda^{Xe}} [1 - \exp(-(\sigma_{a1}^{Xe} \Phi_1 + \lambda^{Xe})t)] \\ &\quad - \frac{\gamma^I \Sigma_{f1} \Phi_1 - \lambda^I I(0)}{\lambda^I - \lambda^{Xe} - \sigma_{a1}^{Xe} \Phi_1} [\exp(-(\sigma_{a1}^{Xe} \Phi_1 + \lambda^{Xe})t) - \exp(-\lambda^I t)], \end{aligned} \quad (6.3.12)$$

where $\gamma \equiv \gamma^I + \gamma^{Xe}$.

Again, we examine what these equations predict about the approach to equilibrium at constant flux and the change in xenon absorption after shutdown from equilibrium operating conditions.

$X(t)$ following Start-Up. If a reactor, initially containing no Xe^{135} or I^{135} , is brought to a flux level Φ_1 at time $t = 0$ and operated at Φ_1 for a long enough time, (6.3.12) shows that, asymptotically, the concentrations of I^{135} and Xe^{135} become

$$\begin{aligned} I(\infty) &= \frac{\gamma^I \Sigma_{f1} \Phi_1}{\lambda^I}, \\ X(\infty) &= \frac{\gamma \Sigma_{f1} \Phi_1}{\sigma_{a1}^{Xe} \Phi_1 + \lambda^{Xe}}. \end{aligned} \quad (6.3.13)$$

Equation (6.3.12) also shows that the speed with which this equilibrium state is reached depends on $\lambda^I = 2.87 \times 10^{-5} \text{ sec}^{-1}$ and on $(\sigma_{a1}^{Xe} \Phi_1 + \lambda^{Xe}) = (2.7 \times 10^{-18} \Phi_1 + 2.09 \times 10^{-5}) \text{ sec}^{-1}$ and thus is about 10^5 sec ($\approx 30 \text{ hr}$) even if Φ_1 is very small. The situation is thus different from the Sm^{149} case, for which the speed of approach to equilibrium is crucially dependent on Φ_1 . Physically the reason for this difference is that Xe^{135} decays radioactively and will thus come into an equilibrium condition, in which the rate of creation of Xe^{135} equals its rate of disappearance, in a time characteristic of both its own half-life and that of I^{135} . If there are neutrons present to add to the destruction rate of Xe^{135} due to radioactive decay, $X(t)$ will approach equilibrium faster. However the 6.7-hour half-life of I^{135} will limit the speed at which equilibrium conditions are reached. Thus 30 hours after start-up is a good estimate of the time to reach equilibrium conditions for any value of Φ_1 .

Another important consequence of the fact that Xe^{135} is radioactive—and another characteristic of xenon behavior that differs from samarium behavior—is that $X(\infty)$ is flux-dependent. The “equilibrium xenon poison” $\sigma_{a1}^{Xe} X(\infty)$ associated with operation at a constant flux level Φ_1 is

$$\sigma_{a1}^{Xe} X(\infty) = \frac{1}{1 + \lambda^{Xe}/(\sigma_{a1}^{Xe} \Phi_1)} \gamma \Sigma_{f1}. \quad (6.3.14)$$

For Φ_1 equal to, say, $10 \text{ neutrons/cm}^2/\text{sec}$, this macroscopic xenon cross section is only about $9 \times 10^{-7} \Sigma_{f1}$. Thus, in a reactor at very low power, Xe^{135} offers a negligible competition for neutrons. However, for $\sigma_{a1}^{Xe} \Phi_1 \approx \lambda^{Xe}$, $\sigma_{a1}^{Xe} X(\infty)$ approaches $\gamma \Sigma_{f1}$ ($= 0.067 \Sigma_{f1}$). For example, for $\Phi_1 = 5 \times 10^{13} \text{ neutrons/cm}^2/\text{sec}$, $\sigma_{a1}^{Xe} X(\infty) = 0.058 \Sigma_{f1}$. The rate of absorption of neutrons by Xe^{135} in a thermal reactor operating at such a flux level is quite comparable to their rate of absorption in the moderator or structural material. As a consequence the presence of “equilibrium xenon” in a high-power thermal

reactor has a significant effect on the critical condition of the reactor. Unless an excess of fuel is provided to overcome the poisoning effects of this isotope, it will not be possible to maintain criticality for more than a few hours of full-power operation.

One of the advantages of fast reactors is that they do not have this problem. The energy-dependent microscopic xenon cross section $\sigma_{a1}^{135}(E)$ is large only in the thermal energy region. Hence the one-group number σ_{a1}^{Xe} will be a million or so times smaller if it is obtained by averaging over a fast-reactor spectrum.

$X(t)$ after Shutdown from Equilibrium Operating Conditions. If we insert (6.3.13) for $I(0)$ and $X(0)$ in (6.3.12) we find that, when a reactor is shut down to $\Phi_1 = 0$ after operating under equilibrium conditions during which the one-group flux at point r has been $\Phi_1(0)$, the I^{135} and Xe^{135} concentrations at r behave according to

$$\begin{aligned} I(t) &= \frac{\gamma^1 \Sigma_{r1} \Phi_1(0)}{\lambda^1} \exp(-\lambda^1 t), \\ X(t) &= \frac{\gamma \Sigma_{r1} \Phi_1(0)}{\sigma_{a1}^{Xe} \Phi_1(0) + \lambda^{Xe}} \exp(-\lambda^{Xe} t) + \frac{\gamma^1 \Sigma_{r1} \Phi_1(0)}{\lambda^1 - \lambda^{Xe}} [\exp(-\lambda^{Xe} t) - \exp(-\lambda^1 t)], \end{aligned} \quad (6.3.15)$$

where, again, time is measured from the instant of shutdown, so that $X(0)$ of (6.3.15) is the equilibrium, preshutdown concentration $X(\infty)$ of (6.3.13).

Equation (6.3.15) shows that $I(t)$ and $X(t)$ both approach zero asymptotically. However the time derivative of $X(t)$ evaluated at $t = 0$ is

$$\left. \frac{\partial X(t)}{\partial t} \right|_{t=0} = \Sigma_{r1} \Phi_1(0) \left(\frac{\sigma_{a1}^{Xe} \gamma^1 \Phi_1(0) - \gamma^{Xe} \lambda^{Xe}}{\sigma_{a1}^{Xe} \Phi_1(0) + \lambda^{Xe}} \right). \quad (6.3.16)$$

Thus, if the equilibrium, preshutdown flux $\Phi_1(0)$ exceeds $\gamma^{Xe} \lambda^{Xe} / \gamma^1 \sigma_{a1}^{Xe}$ ($\approx 3 \times 10^{11}$ neutrons/cm²/sec), the derivative $\partial X(t)/\partial t|_{t=0}$ will be positive, and $X(t)$ will at first grow. In fact, if the initial equilibrium concentration is high, the net amount of xenon present can increase by a factor of two or more before the reactor runs out of I^{135} and the net concentration of xenon begins to decrease. The turnover point depends on the equilibrium conditions at location r but generally occurs about ten hours after shutdown. It may, however, be necessary to wait 40 or 50 hours after shutdown for the decaying concentration $X(t)$ to return to its initial equilibrium value.

Figure 6.6 shows the behavior of Xe^{135} under various transient conditions.

6.4 Accounting for Depletion Effects in Mathematical Models of Reactor Behavior

Fuel burnup and the build-up of nuclei in the reactor resulting from neutron absorption are in principle easily accounted for in the group diffusion-theory model that forms the basis of most nuclear-reactor design computations. One simply determines multigroup

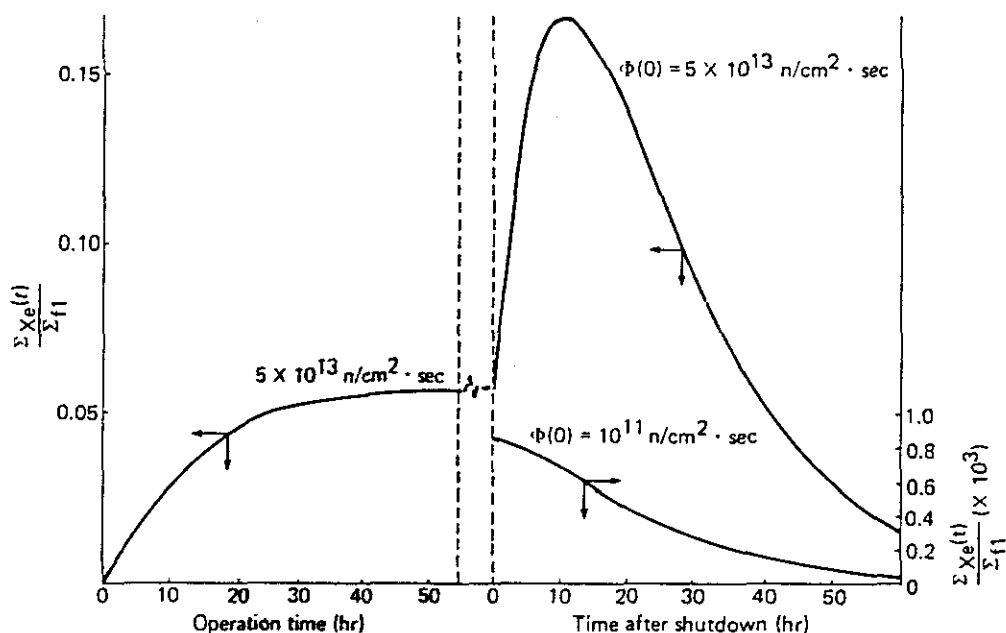


Figure 6.6 Behavior of Xe^{135} in several transient situations.

cross sections throughout the reactor at the beginning of a depletion time step from equations like (5.3.6) (or (5.5.8) if the material is not isotopically homogeneous). These may be used directly in the multigroup diffusion equations of (4.13.7); or they may be used to compute asymptotic spectra, via (5.3.7), for each composition in the reactor so that the spectrum-averaged constants of (5.3.8) and (5.3.9) may be found for use in few-group diffusion-theory calculations. They, by solving these group diffusion equations, we can determine the spatial dependence of the flux for each group throughout the reactor. These flux shapes are assumed to remain constant throughout each time step, and changes in the concentrations of the important nuclei are found by solving equations such as (6.2.6) or (6.2.8), with the one-group reaction-time constants $\sigma_i^j \Phi_i$ replaced by the corresponding few-group or multigroup expressions $\sum_g \sigma_{ig}^j \Phi_g$. With the new material concentrations determined in this manner for the end of the depletion time step, new energy-group parameters are found, and the tandem depletion process is continued.

In practice a great deal of ingenuity is necessary in making this procedure economically feasible. Problems associated with searching for critical conditions, with keeping track of the number densities of all the time-dependent isotopes throughout the reactor, and with using large time intervals for the depletion calculations, all require that special strategies be developed if the running time for a depletion calculation is not to become unaccept-